Tutorial on Integer Programming for Visual Computing

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1 Notation

- The vector space is denoted as $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{m\times n}, \mathbb{V}, \mathbb{W}$
- Matricies are denoted by upper case, italic, and boldface letters: $\boldsymbol{A}_{m\times n}$
- Vectors are column vectors denoted by boldface and lower case letters: $\mathbf{x} \in \mathbb{R}^{n\times 1}$
- $\mathbb{1}_n \in \mathbb{R}^n$ is a $n \times 1$ vector of all ones
- $\boldsymbol{I}_n$ is $n \times n$ identity matrix.
- $\mathbf{e}_i$ is the unit vector where only the i-th element is 1 and the rest are 0.

2 Optimization Terms

- General Form

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) \\ s.t \quad g_i(\mathbf{x}) \leq b_i, \quad & 1 \leq i \leq m \\ & \mathbf{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \end{aligned}$$

- Details:
  - $\mathbf{x}$ is a vector of $n = n_1 + n_2$ variables
  - g_i are called constraint functions
  - f is called objective function
- The feasible region is:

$$F = \{\mathbf{x} \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} | g_i(\mathbf{x}) \leq b_i\}$$

- A solution is an assignment of values to variables
- An optimal solution $\mathbf{x}^*$ has smallest value of f among all feasible solutions.
- term optimization vs. term programming

3 Linear Programming

3.1 General Form

- General form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T \mathbf{x} \\ \boldsymbol{A}\mathbf{x} &\leq \mathbf{b} \end{aligned}$$

- $\mathbf{x} \in \mathbb{R}^n$ is a vector of variables
- $\mathbf{c} \in \mathbb{R}^n$ is a vector of known coefficients (weights)
- $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is a matrix. Each of the m rows of the matrix defines the coefficients of a linear inequality.
- $\mathbf{b} \in \mathbb{R}^m$ is a vector. Each entry b_i is on the right hand side of inequality i.

3.2 Example

- Example with two variables and two constraints:

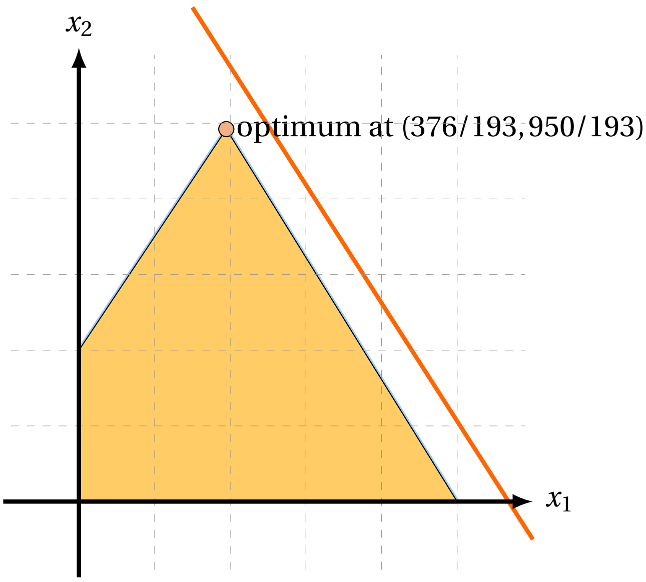
$$\begin{aligned} \min_{x_1, x_2} \quad & c_1 x_1 + c_2 x_2 \\ a_{11} x_1 + a_{12} x_2 &\leq b_1 \\ a_{21} x_1 + a_{22} x_2 &\leq b_2 \end{aligned}$$

- More specific example with two variables and two constraints:

$$\begin{aligned} \min_{x_1, x_2} \quad & -4x_1 - 2x_2 \\ x_1 + 2.4x_2 &\leq 12.1 \\ 7x_1 &\leq 22 \end{aligned}$$

- Graphical Example:

$$\begin{aligned} \max_{x_1, x_2} \quad & 100x_1 + 64x_2 \\ 50x_1 + 31x_2 &\leq 250 \\ 3x_1 - 2x_2 &\geq -4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$



3.3 How to solve linear programming problems?

- No analytic formula for the solution
- Reliable and efficient algorithms and software, e.g.
  - Simplex algorithm
  - Interior point algorithms
- Computation time proportional to n^2m if $m \geq n$; less with structure
- Formulating a problem as linear programming problem is already non-trivial

3.4 From linear programming to linear integer programming

- Optimization problem:

$$\min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x}$$
$$A\mathbf{x} \leq \mathbf{b}$$

- floating point variables
  - $\mathbf{x} \in \mathbb{R}^n$
  - linear program (LP)
- integer variables
  - $\mathbf{x} \in \mathbb{Z}^n$
  - (linear) integer program (IP)
- binary variables

  - $\mathbf{x} \in \{0,1\}^n$
- float and integer variables
  - $\mathbf{x}$ is split into two groups of variables, $\mathbf{x_I}$ and $\mathbf{x_F}$
  - $\mathbf{x_F} \in \mathbb{R}^{n_1}$ and $\mathbf{x_I} \in \mathbb{Z}^{n_2}$
  - mixed integer program (MIP)

3.5 Variations of the standard form

- Optimization problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ & \boldsymbol{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

- switch min and max
- switch $\leq$ and $\geq$
- include constraints with $=$ as separate category
- require all variables to be positive (≥ 0)
- Example Optimization problem:

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ & \boldsymbol{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

3.6 Comments about formulations

Definition 1. A polyhedron P is a subset of $\mathbb{R}^n$ described by a finite set of linear constraints. $P = \{x \in \mathbb{R}^n : \boldsymbol{A}\mathbf{x} \leq \mathbf{b}\}$

Definition 2. A polyhedron $P \subseteq \mathbb{R}^{n_1+n_2}$ is a formulation for a set $X \subseteq \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$ if and only if $X = P \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2})$.

Definition 3. A convex combination of points from a set S, $x_1, x_2, ..., x_k \in S$, is any point of form $\theta_1 x_1 + \theta_2 x_2 + ... + \theta_k x_k$, where $\theta_i \geq 0, i = 1...k, \sum_{i=1}^{k} \theta_i = 1$. A set S is convex iff any convex combination of points in S is in S.

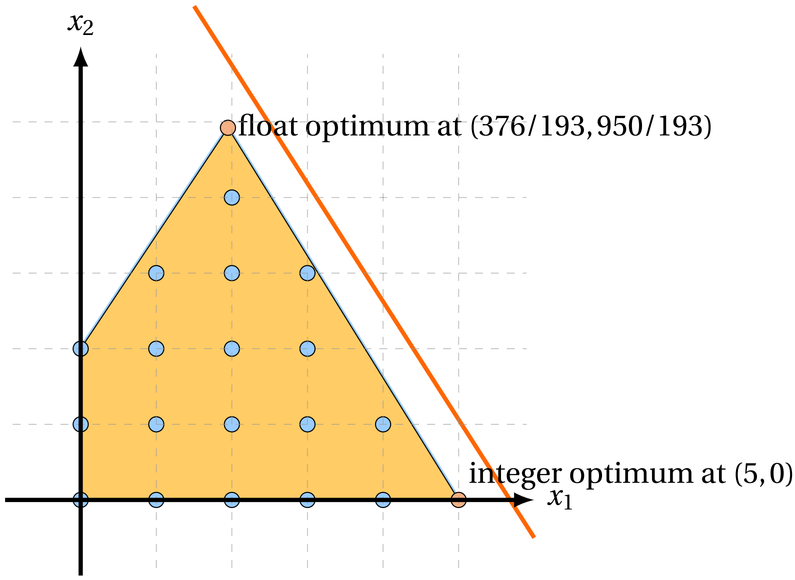
Definition 4. The convex hull $conv\ S$ is the set of all convex combinations of points in S

- The formulation has to enclose all feasible integer points, but no infeasible integer points
- Runtime depends on
  - number of variables
  - number of constraints

  - tightness of fit
- Formulation A is at least as strong as B if $A \subseteq B$
- Formulation A is stronger than B if $A \subset B$
- A formulation A is ideal if $conv($ feasible solutions $) = A$

3.7 Graphical Example

$$\begin{aligned}
\max_{x_1,x_2} \quad & 100x_1 + 64x_2 \\
50x_1 + 31x_2 &\leq 250 \\
3x_1 - 2x_2 &\geq -4 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
x_1, x_2 &\in \mathbb{Z}
\end{aligned}$$



- Rounded solution might not be feasible
- Rounded solution might be far from optimal solution

3.8 Different Components of Optimization in the literature

- Modeling:
  - How to formulate an application problem as a standard optimization problem?
- Algorithm Development:
  - How to derive new optimization algorithms for standard optimization problems?
  - How to derive new optimization algorithms for specialized optimization problems?
- Optimization Theory:

- Finding convergence guarantees, bounds, ... of optimization algorithms

3.9 Different Components of Optimization in Visual Computing

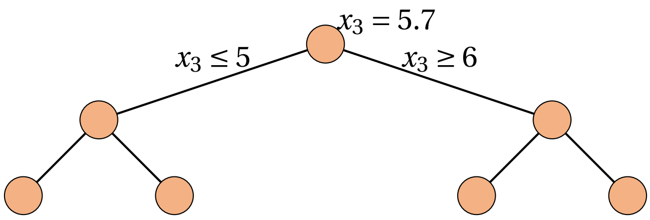
- Modeling:
  - propose an interesting problem formulation for a new or an existing problem in visual computing?
- Algorithm Development:
  - propose a new algorithm for a specific optimization problem in visual computing
- Modeling + Algorithm Development
- Theory
  - typically not done in visual computing, but in optimization and machine learning

3.10 How to solve an IP Problem?

- use a standard solver such as Matlab, Gurobi, Mosek, ... and see what happens
- create a new heuristic solver

3.11 Branch and Bound

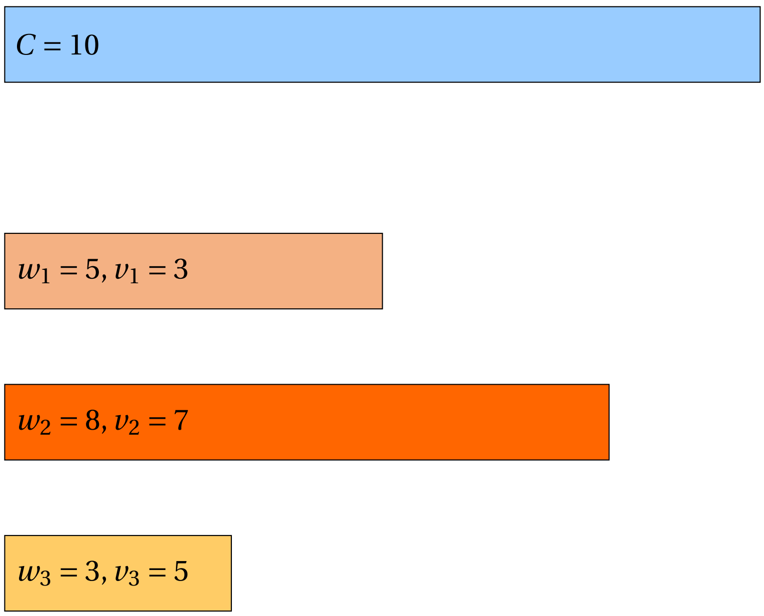
- How to create upper and lower bounds for (the objective value of) the solution?
  - The LP relaxation is a lower bound for the optimal solution
  - Any particular feasible solution is an upper bound for the optimal solution
- If we solve the LP relaxation of an MILP problem we distinguish 3 cases:
  - LP is infeasible → MILP is infeasible
  - Optimal LP solution is feasible solution for MILP problem → optimal solution
  - LP is feasible and optimal LP solution is not feasible for MILP → lower bound
- First two cases we are finished, third case we branch (recursively)
- The most common way to branch is to do the following
  - Select a variable i whose value $\hat{x}_i$ is fractional in the LP solution
  - Create two subproblems:
    - Add constraint $x_i \leq \lfloor \hat{x}_i \rfloor$
    - Add constraint $x_i \geq \lceil \hat{x}_i \rceil$



4 Example Problems

4.1 Knapsack Problem

- Input:
  - a set of items i with values v_i and weights w_i
  - a knapsack with maximum capacity c
- Goal: pack a subset of items into the knapsack, such that
  - the sum of weights does not exceed the capacity C
  - the sum of the values is maximized
- Example



- Formulation:
  - variables: $x_i = 1$ means we pack item i

–

$$\min_{\mathbf{x}} \quad \mathbf{v}^T\mathbf{x}$$
$$\mathbf{w}^T\mathbf{x} \le c$$
$$x_i \in 0, 1$$

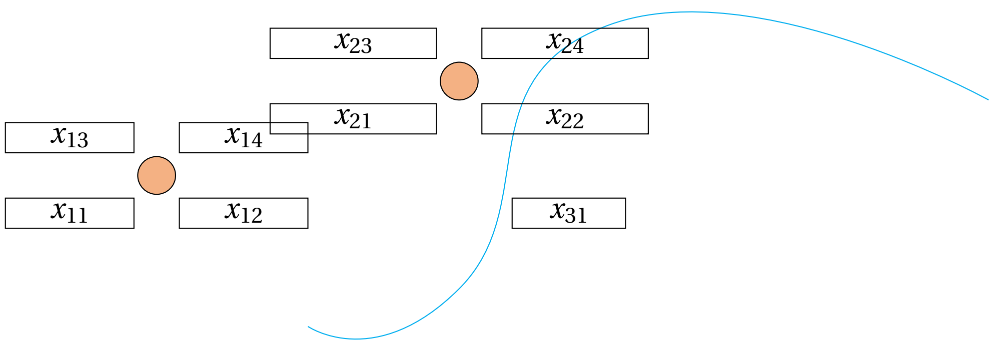
- Difficulty:
  - NP-hard
  - (pseudo-polynomial) Dynamic Programming solution exists for integer weights and capacity.

4.2 Matlab Code

```
C = 750
weights = [70; 73; 77; 80; 82; 87; 90; 94; 98; 106; 110; 113; 115; 118; 120];
values = [135; 139; 149; 150; 156; 163; 173; 184; 192; 201; 210; 214; 221; 229;
240];
LZero = zeros(length(weights),1);
LOne = ones(length(weights),1);
LCount = 1:length(weights);
tic;
intlinprog( -values, LCount, weights', C, [], [], LZero, LOne)
toc;
```

4.3 Map Labeling

- Input:
  - a set of map objects i where each object has a discrete set of possible label positions j
  - costs $\mathbf{c}$ for each label placement
- Goal: place at least one label per object without overlap
- Illustration: two cities one river



- Variables
  - $x_{ij} = 1$ if label for object i is placed at position j
- Constraints:
  - Binary constraints:

$$x_{ij} \in \{0,1\}$$

  - Coverage constraint - each element is labeled exactly once:

$$\forall i \quad \sum_j x_{ij} = 1$$

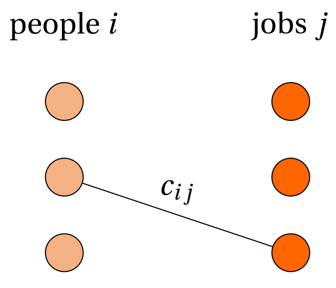
  - Non-overlap for conflicting placements:
    - for each pair of overlapping placements ij and lm

$$x_{ij} + x_{lm} \leq 1$$

- Objective: $\min \sum_i \sum_j c_{ij} x_{ij}$

4.4 Assignment Problem

- Input:
  - n people to carry out n jobs
  - c_{ij}: cost of assigning person i to job j
- Goal: assign each person to exactly one job, so that each job has one person assigned to it.
- Illustration:



- Variables
  - $x_{ij} = 1$ if person i is assigned to job j
- Objective:

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

- Constraints:
  - Binary constraints:

$$x_{ij} \in \{0,1\}$$

– Limited work: each person i does exactly one job

$$\forall i \quad \sum_j x_{ij} = 1$$

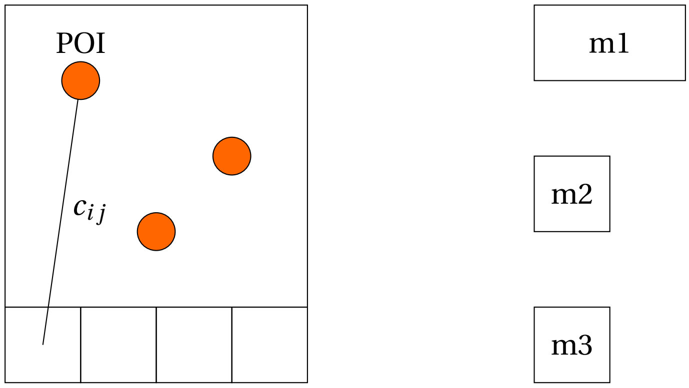
– Coverage constraint - each job is done by one person:

$$\forall j \quad \sum_i x_{ij} = 1$$

- Difficulty:
  – Hungarian Method (Kuhn–Munkres algorithm or Munkres assignment algorithm)
  – Auction algorithm

4.5 Tourist Map Layout

- Input:
  – overview map with Points of Interest (POIs)
  – detail maps for each POI
  – positions for detail maps
  – costs c_{ij} for assigning POI i detail map position j
- Goal: assign each detail map to one position.
- Illustration:



- Variables
  – $x_{ij} = 1$ if map i is assigned to position j
- Objective:

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

- Constraints:
  – Binary constraints:

$$x_{ij} \in \{0, 1\}$$

– Each map i is assigned once

$$\forall i \quad \sum_j x_{ij} = 1$$

– No overlap between maps:

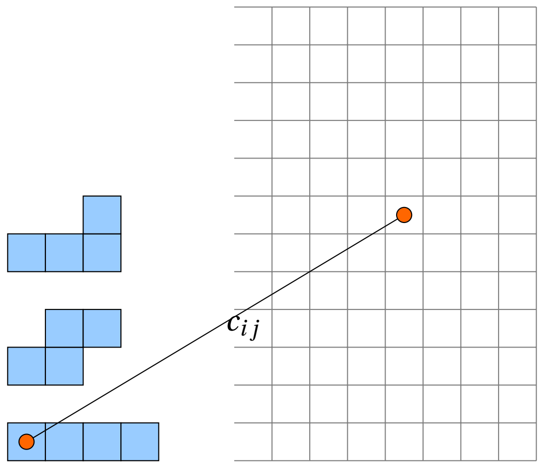
$$\forall j \quad \sum_{(i,j)\in O_j} x_{ij} = 1$$

  ◦ O_j is the set of all placements that overlap position j

- Literature: Birsak et al., "Automatic Generation of Tourist Brochures", Eurographics 2014.

4.6 Tiling

- Input:
  – a set of tiles i
  – a domain consisting of positions j
  – costs c_{ij} for assigning tile i to position j
  – minimum and maximum number of times tile i is allowed to be used (min_i, max_i)
- Goal: cover the domain with the given tiles
- Illustration:



- Variables
  – $x_{ij} = 1$ if leftmost square of tile i is assigned to position j
- Objective:

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

- Constraints:

– Binary constraints:

$$x_{ij} \in \{0, 1\}$$

– Each tile i is assigned between its within its allowed limits

$$\forall i \quad min_i \leq \sum_j x_{ij} \leq max_i$$

– No overlap between squares in the domain:

$$\forall j \quad \sum_{(i,j)\in O_j} x_{ij} = 1$$

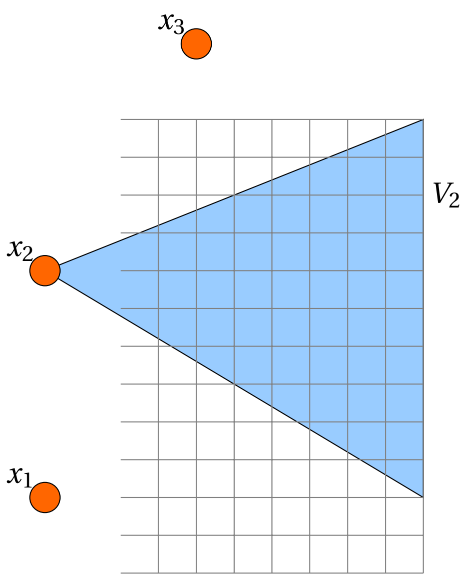
◦ O_j is the set of all tile placements that overlap position j

4.7 Shape Matching

- Input:
  – two shapes where each shape has n vertices.
  – a cost c_{ij} for assigning vertex i from shape 1 to vertex j on shape 2,
- Goal: assign each vertex on shape 1 to exactly one vertex on shape 2
- Formulation: identical to the assignment problem
- Literature:
  – Vestner et al., "Product Manifold Filter: Non-Rigid Shape Correspondence via Kernel Density Estimation in the Product Space", CVPR 2017.

4.8 Camera Placement

- Input:
  – a domain sampled into positions p
  – a set of possible camera positions i
- Goal: select a minimal set of cameras that cover the domain
- Illustration:



- Variables
  - $x_i = 1$ if camera position i is selected
- Objective:

$$\min \sum_i x_i$$

- Constraints:
  - Binary constraints:

$$x_i \in \{0, 1\}$$

  - Position conflict constraints

$$\forall i \quad \sum_{j \in N_i} x_j \leq 1$$

  - N_i is the set of locations that conflict with location i
  - Visibility constraint:

$$\boldsymbol{V}\mathbf{x} \geq 1$$

    - the i^{th} column of $\boldsymbol{V}$ is a binary mask that encodes what positions are seen by camera i

4.9 Graph Review

- Graph (V, E)
  - V is a set of nodes
  - E is a set of edges
- $E(S) = \{e = (i, j) : i, j \in S\}$

- $\delta(S) = \{e = (i,j) : i \in S \text{ and } j \in V \setminus S\}$
- $\delta(i)$ are all edges incident to node i.
- A tree is a connected graph with $|V| - 1$ edges.

4.10 Minimum Spanning Tree

- Input:
  - a graph (V, E)
  - the cost c_e for selecting edge $e \in E$.
- Goal: find a minimum cost spanning tree
- Variables
  - $x_e = 1$ if edge e is selected
- Binary constraints:

$$x_e \in \{0, 1\}$$

- Number of edges constraint:

$$\sum_{e \in E} x_e = n - 1$$

- Cut constraint:

$$\forall S \subset V, S \neq \emptyset, V \qquad \sum_{e \in \delta(S)} x_e \geq 1$$

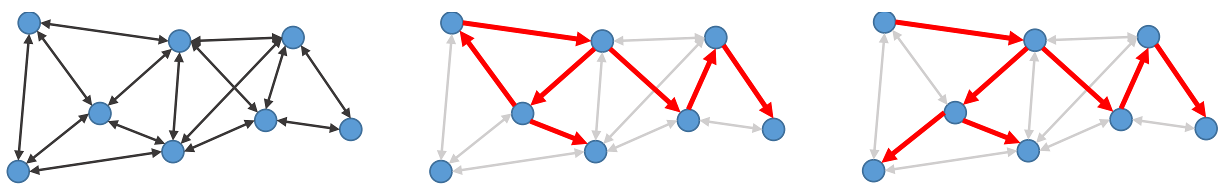
- Objective function:

$$\min \sum_{e \in E} c_e x_e$$

- We call the linear relaxation of this formulation P_{cut}
- Alternative constraint: subtour elimination constraint

$$\forall S \subset V, S \neq \emptyset, V \qquad \sum_{e \in E(S)} x_e \leq |S| - 1$$

- We call the resulting linear relaxation of the formulation P_{sub}
- Notes:
  - P_{sub} is the convex hull of the set of feasible solutions.
  - P_{sub} is a strictly better formulation than P_{cut}.



4.11 Traveling Salesman

- Input:
  - a graph (V, E)
  - the cost c_e for selecting edge $e \in E$.
- Goal: find a minimum cost tour
- Variables
  - $x_e = 1$ if edge e is selected
- Binary constraints:

$$x_e \in \{0, 1\}$$

- Number of incident edges constraint:

$$\forall i \quad \sum_{e \in \delta(i)} x_e = 2$$

- Cut constraint:

$$\forall S \subset V, S \neq \emptyset, \quad \sum_{e \in \delta(S)} x_e \geq 2$$

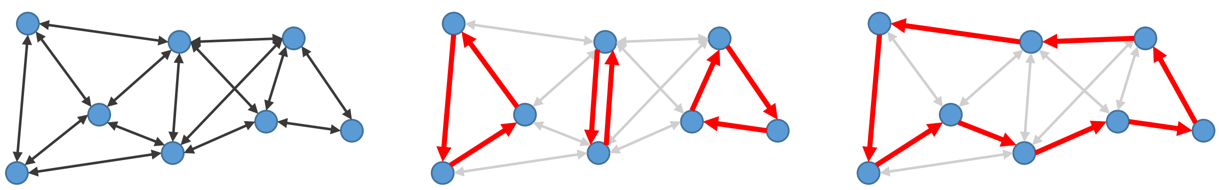
- Objective function:

$$\min \sum_{e \in E} c_e x_e$$

- Alternative constraint: subtour elimination constraint

$$\forall S \subset V, 2 \leq |S| \leq |V| - 1 \quad \sum_{e \in E(S)} x_e \leq |S| - 1$$

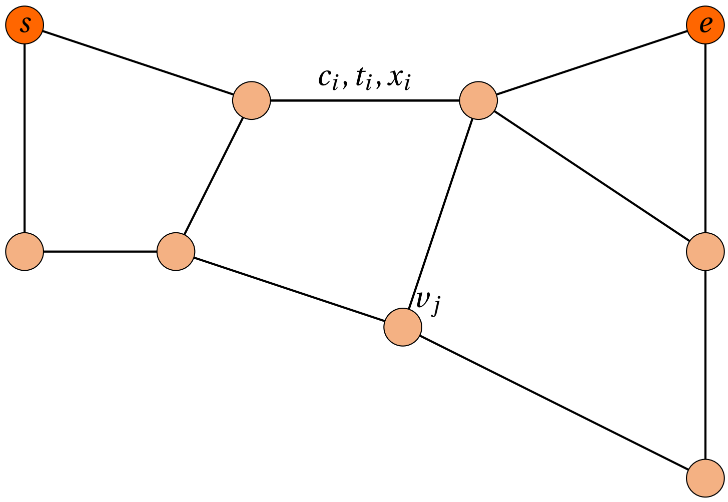
- Similarly, we call the resulting linear relaxations P_{cut} and P_{sub}
  - $P_{cut} = P_{sub}$
  - Neither is the convex hull of the feasible points



4.12 City Exploration

- Input:
  - a city map as graph (V, E)
  - $\mathbf{c} \in \mathbb{R}^{|E|}$ - the attractiveness of each edge

- $\mathbf{t} \in \mathbb{R}^{|E|}$ - time it takes to walk along an edge
  - T - maximum time for the walk
  - a designated start node s and end node e
- Goal: find a walk through the city from from start node to end node that explores the most attractive edges but stays under the time limit.
- Illustration



- Variables
  - $x_i = 1$ if edge i is selected
  - $v_j = 1$ if vertex j is selected
- Binary constraints:

$$x_i, v_j \in 0, 1$$

- Time constraint:

$$\mathbf{t}^T \mathbf{x} \leq T$$

- Connection constraint:

$$\sum_{i \in N_j} x_i = v_j \quad \sum i \in N_s x_i = 1 \quad \sum_{i \in N_e} x_i = 1$$

  - N_j is the set of edges incident to vertex j
- Objective function:
  - $\max \mathbf{c}^T \mathbf{x}$
- Cycles:
  - the formulation can create closed cycles
  - solution 1: lazy constraint adding
  - solution 2: add constraints that forbid cycles (similar to MST and TS formulations)

5 MIP Modeling Techniques

5.1 AND of variables

- "y is true if all elements in $\mathbf{x}$ are true. y is false otherwise.":

$$y = x_0 \wedge x_1 \wedge ... \wedge x_{N-1}$$

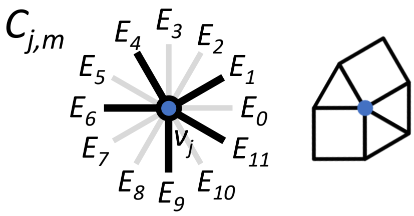
- y and $\mathbf{x}$ are Boolean variables. x_0, x_1, ..., x_{N-1} are the elements in $\mathbf{x}$. N is the size of $\mathbf{x}$.
- Trivial way to model:

$$y = x_0 x_1 ... x_{N-1}$$

  It is not going to work!
- As linear inequalities:

$$0 \leq \sum \mathbf{x} - Ny \leq N-1$$

- Example:
  - Vertex configurations in a 2D triangle-quad hybrid mesh:



$C_j m$ is the m-th configuration for vertex v_j. $C_j m$ contains E_1, E_4, E_6, E_9, and E_{11} out of v_j's twelve adjacent edges:

$$C_j m = !E_0 \wedge E_1 \wedge !E_2 \wedge !E_3 \wedge E_4 \wedge !E_5 \wedge E_6 \wedge !E_7 \wedge !E_8 \wedge E_9 \wedge !E_{10} \wedge E_{11}$$

As linear inequalities:

$$0 \leq (1-E_0)+E_1+(1-E_2)+(1-E_3)+E_4+(1-E_5)+E_6+(1-E_7)+(1-E_8)+E_9+(1-E_{10})+E_{11}-12y \leq 11$$

5.2 OR of variables

- "y is true if any element in $\mathbf{x}$ is true. y is false otherwise.":

$$y = x_0 \vee x_1 \vee ... \vee x_{N-1}$$

- As linear inequalities:

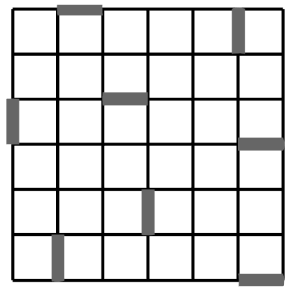
$$-N+1 \leq \sum \mathbf{x} - Ny \leq 0$$

- Example:
  - Converge constraint: a vertex is "covered" if and only if at least one of the edges that are within a close proximity is selected.

$$v_i = e_0 \vee e_1 \vee ... \vee e_{N-1}$$

v_i is the Boolean variable indicating if the vertex is covered. e_0, e_1, ..., e_{n-1} are Boolean variables of edges within a close proximity to the vertex.

◦ For a minimal-vertex cover problem, we may require that the coverage variables of all vertices are true while minimizing the number of selected edges.



5.3 XOR of variables

- "y is true if elements in **x** sum to odd. y is false if elements in **x** sum to even."

$$y = x_0 \oplus x_1 \oplus ... \oplus x_{N-1}$$

- As linear inequalities:

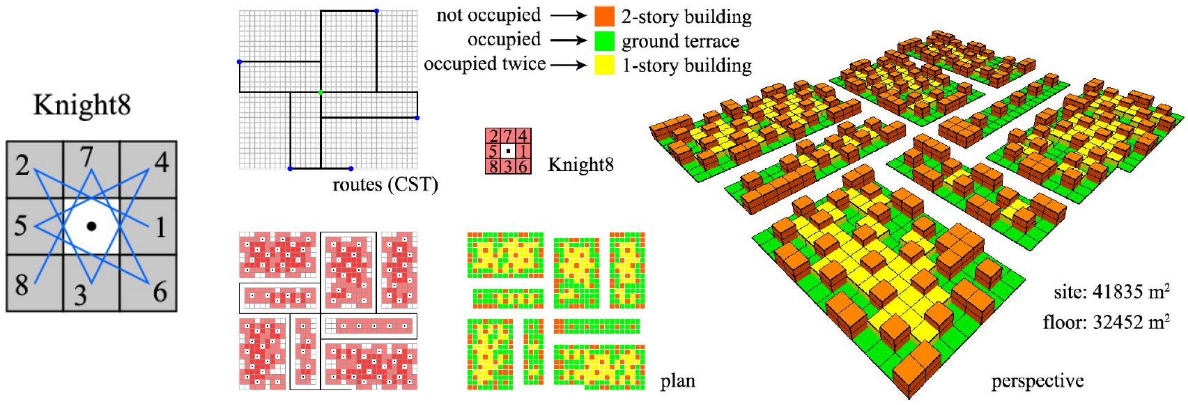
$$y = x_0 + x_1 + ... + x_{N-1} - 2t$$

  t is an integer slack variable. $0 \le t \le N-1$.
- Alternatively, model it as a sequence of 2-inputs XORs (the t variables become Booleans).

5.4 Special order set (SOS)

- Special Ordered Sets of type 1 (SOS1):
  - Given an ordered set of variables, **q**, at most one element in **q** can be non-zero.
- Special Ordered Sets of type 2 (SOS2):
  - Given an ordered set of variables, **q**, at most two elements in **q** can be non-zero. And if two elements are non-zero, they must be consecutive in their ordering.
- Supported by popular MIP solvers such as Gurobi and IBM CPLEX. These solvers use special branching strategies to take advantage of SOSs.
- Examples:
  - A SOS1 set, **x**, of Boolean variables x_0, x_1, ..., x_{N-1}, means that:

$$x_0 + x_1 + ... + x_{N-1} \le 1$$



- SOS2: "knight8" template for translational symmetry in urban layout design:
  - Integer programming for urban design. Hao Hua, Ludger Hovestadt, Peng Tang, and Biao Li. European Journal of Operational Research (EJOR), 2018.

5.5 Exhaustive enumeration of all feasible solutions of a (Boolean) IP problem

- Let **Z** denotes a feasible solution of a IP problem with only Boolean variables. We can forbid **Z** to be feasible, that is,

$$\mathbf{Z} \wedge F = \emptyset$$

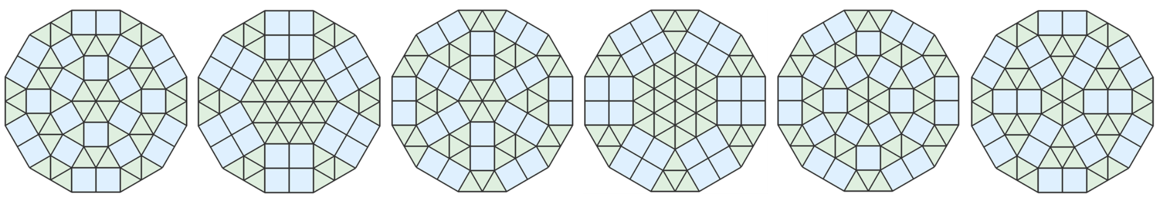
  where F is the feasible region of the problem, by adding the following constraint:

$$\sum_{0 \le i \le N-1} (x_0 \text{ if } \; Z_i \text{ is true, or } (1 - x_i) \text{ if } Z_i \text{ is false}) \le N - 1$$

  to the IP formulation. **x** denotes the variables. N is the number of variables.
- An enumeration of unique feasible solutions can be done by repeatedly solving the IP problem with all previously retrieved solutions forbidden.
- An exhaustive enumeration proceeds until the problem becomes infeasible.
- Examples:
  - Given a IP with three Boolean variables, x_0, x_1, and x_2, adding the following constraint would forbid $(0, 1, 0)$ as a feasible solution:

$$(1 - x_0) + x_1 + (1 - x_2) <= 2$$

- Exhaustive enumeration of triangle-quad tilings in a 12-gon with side length 2.



5.6 Big-M method

- Use Boolean slack variables with sufficiently large coefficients to allow constraints to be "de-activated".
- That is, rewriting a linear constraint:

  $$a^T\mathbf{x} \leq b$$

  to be:

  $$a^T\mathbf{x} \leq b + My$$

  would allow it to be violated. M is a sufficiently large positive constant and y is a Boolean slack variable. When it is violated, y is true.
- Optionally, add y to the objective function (to minimize) to introduce penalty for the constraints to be violated.
- Example:
  - "Constrain the union of two (mutually exclusive) constraints to be true":

    $$a_0^T\mathbf{x_0} \leq b_0 \quad \text{or} \quad a_1^T\mathbf{x_1} \geq b_1$$

    - As linear inequalities:

      $$a_0^T\mathbf{x_0} \leq b_0 + M(1-y)$$
      $$a_1^T\mathbf{x_1} \geq b_1 - My$$

      where M is a sufficiently big positive constant and y is a Boolean slack variable.
    - Example:

      $$x \leq 2 \quad \text{or} \quad x \geq 6$$

      is reformulated as:

      $$x \leq 2 + M(1-y),$$
      $$x \geq 6 - My$$

- Discussions
  - Many modeling techniques in MIP are variations of the big-M method.
  - In general, big-M methods are more preferable than the equivalent non-linear formulations.
  - M should be kept as small as possible. Very big M impacts performance.
- Literature:
  - Indicator Constraints in Mixed-Integer Programming. Andrea Lodi, Amaya Nogales-Gómez, Pietro Belotti, Matteo Fischetti, Michele Monaci, Domenico Salvagnin, and Pierre Bonami. SCIP Workshop 2014.

- Integer Programming Formulations 2. James Orlin. Course notes of Optimization Methods in Management Science on MIT OCW.

6 Quadratic Programming

6.1 General Form

- General form:

$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^T \boldsymbol{Q}\mathbf{x} + \mathbf{c}^T\mathbf{x}$$
$$\boldsymbol{A}\mathbf{x} \leq \mathbf{b}$$

- $\mathbf{x} \in \mathbb{R}^n$ is a vector of variables
- $\mathbf{c} \in \mathbb{R}^n$ is a vector with known entries
- $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ is a symmetric matrix with known entries
- $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is a matrix. Each of the m rows of the matrix define the coefficients of a linear inequality.
- $\mathbf{b} \in \mathbb{R}^m$ is a vector. Each entry b_i is on the right hand side of inequality i.

6.2 Comments

- if $\boldsymbol{Q} \succ 0$ (the matrix is positive-definite) the optimization is convex

7 Quadratic Integer Programming Examples

7.1 Quadratic Assignment

- Input:
  - a set of n facilities i
  - a set of n possible facility location j
  - costs $c_i jkl$ for assigning facilty i to location j and facility k to location l
- Goal: assign facilities to grid cells to minimize costs
- Variations:
  - costs $c_i jkl$ can be modeled arbitrarily
  - costs $c_i jkl$ are modeled as the product $c_i jkl = f_i k d_j l$, where $f_i k$ is a flow between facility i and k and $d_j l$ is a distance between j and l. This is the classical quadratic assignment problem.

- Variables
  - $x_{ij} = 1$ if facility i is assigned to location j
- Objective:

$$\min \sum_i^n \sum_j^n \sum_k^n \sum_l^n c_{ijkl} x_{ij} x_{kl}$$

- Constraints:
  - Binary constraints:

$$x_{ij} \in \{0,1\}$$

  - Non-overlap: each facility i has exactly one position

$$\forall i \quad \sum_j x_{ij} = 1$$

  - Coverage: each position is covered by exactly one facility

$$\forall j \quad \sum_i x_{ij} = 1$$

- Literature: Loiola et al., "A survey for the quadratic assignment problem", European Journal of Operational Research 2007.

7.2 Quadratic Assignment for Images

- Input:
  - a set of n images with image distances d_{ij}
  - a set of n possible image positions with distances g_{kl}
  - costs $c_{ijkl} = f(d_{ik}, g_{jl})$
- Goal: assign images to grid cells to minimize the costs
- Variables
  - $x_{ij} = 1$ if image i is assigned to grid cell j
- Objective:

$$\min \sum_i^n \sum_j^n \sum_k^n \sum_l^n c_{ijkl} x_{ij} x_{kl}$$

- Constraints:
  - Binary constraints:

$$x_{ij} \in \{0,1\}$$

  - Non-overlap: each image i has exactly one position

$$\forall i \quad \sum_j x_{ij} = 1$$

- Coverage: each position is covered by exactly one image

$$\forall j \quad \sum_i x_{ij} = 1$$

- Literature: Fried et al., "IsoMatch: Creating Informative Grid Layouts", Eurographics 2015.

7.3 Quadratic Assignment for Shape Matching

- Literature:
  - Dym et al., *DS++: A Flexible, Scalable and Provably Tight Relaxation for Matching Problems*, ACM TOG 2017.
  - Kezurer et al., *Tight Relaxation of Quadratic Matching*, SGP 2015.

7.4 Joint Segmentation

- Input:
  - Two shapes. Each shape is subdivided into smaller patches P_1 and P_2, respectively
  - A set of candidate segments for each shape: S_1 and S_2. Each segment consists of multiple patches.
  - A cost vector $\mathbf{c}$ where $\mathbf{c_{ij}}$ is the cost selecting a segment j in shape i.
  - A cost vector $\boldsymbol{d}$ where d_{ij} encodes the cost of mapping segment i in shape one to segment j in shape two.
  - A cost matrix $\boldsymbol{Q}$ where q_{ijkl} encodes the cost of mapping segment i in shape one to segment j in shape two and segment k in shape one to segment l in shape two.
- Variables:
  - $x_{ij} = 1$ if segment j is selected from shape i.
  - $p_{ij} = 1$ if patch j is selected from shape i.
  - m_{ij} if segment i in shape one maps to segment j in shape two.
- Literature:
  - Huang et al., *Joint-Shape Segmentation with Linear Programming*, ACM TOG 2011.

7.5 Fit and Diverse Sampling

8 Quadratically Constrained Quadratic Programming

8.1 General Form

- General form:

$$
\begin{aligned}
\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^T \boldsymbol{Q}_{\mathbf{0}}\mathbf{x} + \mathbf{c_0}^T\mathbf{x} \\
\mathbf{x}^T \boldsymbol{Q}_i\mathbf{x} + \mathbf{c_i}^T\mathbf{x} \le b_i
\end{aligned}
$$

- $\mathbf{x} \in \mathbb{R}^n$ is a vector of variables
- $\mathbf{c_i} \in \mathbb{R}^n$ are vectors with known entries
- $\boldsymbol{Q}_i \in \mathbb{R}^{n\times n}$ are symmetric matrices with known entries
- $\mathbf{b} \in \mathbb{R}^m$ is a vector. Each entry b_i is on the right hand side of inequality i.

8.2 Mixed Integer Quadratically Constrained Programming

- Can be solved by commercial solvers